Roark's Formulas for Stress & Strain, 7th Edition.
Chapter 8: Beams; Flexure of Straight Bars
8.3 Three-Moment Equation

EXAMPLE 8.3.1

Disclaimer

User Notices

Consider two contiguous spans loaded as shown in Fig. 1. In addition to loading shown, it is known that the left end of span 1 had settled $y_2 - y_1$ relative to the right end of the span, and similarly that the span 2 has settled an amount $y_3 - y_2$ relative to the right end. (Note and y_1, y_2, y_3, are considered positive upward as usual.) The individual loadings are shown in **Fig. 2 (a)** and **Fig. 2 (b)**. Determine the relationship of the applied loads and the moment at the intermediate support.

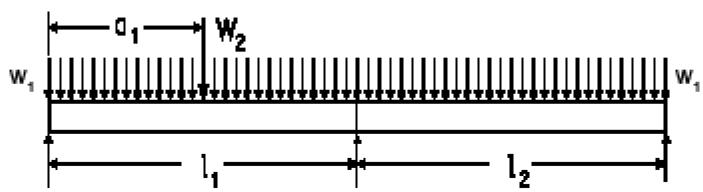


Figure 1

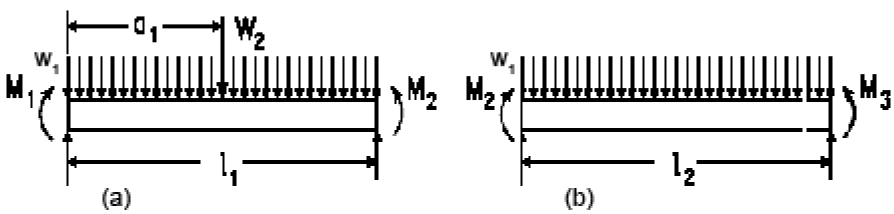


Figure 2

Geometry:

$L_1 := 10 \cdot ft$

$y_1 := 0 \cdot in$

$L_2 := 10 \cdot ft$

$y_2 := 0 \cdot in$

$a_1 := 0.405 \cdot L_1$

$y_3 := 0 \cdot in$

Loads & Constrains:

$w_1 := 84.75 \cdot \frac{lbf}{ft}$

$W_2 := 520 \cdot lbf$

Material Properties:

$\sigma_{ty} := 25 \cdot ksi$

$\sigma_{alw} := 0.5 \cdot \sigma_{ty}$

Solution

Find the slope at the right end of span 1.
From the **case 8.1.1e**:

$$\theta_{2B1} = \frac{W_2 \cdot a_1}{6 \cdot E \cdot I \cdot L_1} \cdot \left(L_1^{\,2} - a_1^{\,2}\right)$$

or

$$\theta_{2B1} = \frac{1}{E \cdot I} \cdot \frac{W_2 \cdot a_1}{6 \cdot L_1} \cdot \left(L_1^{\,2} - a_1^{\,2}\right)$$

From the **case 8.1.2e**:

$$\theta_{2B2} = \frac{w_a}{24 \cdot E \cdot I \cdot L_1} \cdot \left(L_1^{\,2} - a^2\right)^2 + \frac{w_L - w_a}{360 \cdot E \cdot I \cdot L_1} \cdot \left(L_1 - a\right)^2 \cdot \left(8 \cdot L_1^{\,2} + 9 \cdot a \cdot L_1 + 3 \cdot a^2\right)$$

where

$$a = 0 \cdot ft$$

$$w_a = w_L = w_1$$

thus

$$\theta_{2B2} = \frac{1}{E \cdot I} \cdot \frac{w_1 \cdot L_1^{\,3}}{24}$$

From the **case 8.1.3e**:

$$\theta_{2B3} = \frac{M_1}{6 \cdot E \cdot I \cdot L_1} \cdot \left(L_1^{\,2} - 3 \cdot a^2\right)$$

where

$$a = 0 \cdot ft$$

thus

$$\theta_{2B3} = \frac{1}{E \cdot I} \cdot \frac{M_1 \cdot L_1}{6}$$

and

$$\theta_{2B4} = \frac{-M_2}{6 \cdot E \cdot I \cdot L_1} \cdot \left(L_1^{\,2} - 3 \cdot a^2\right)$$

where

$$a = L_1$$

thus

$$\theta_{2B4} = \frac{1}{E \cdot I} \cdot \frac{M_2 \cdot L_1}{3}$$

Using the **cases 1e, 2e** and **3e** from the **Table 8.1** and noting the relative deflections mentioned above, the expression for the slope at the right end of span 1 is

$$\theta_2 = \frac{1}{E \cdot I} \cdot \frac{W_2 \cdot a_1}{6 \cdot L_1} \cdot \left(L_1^{\,2} - a_1^{\,2}\right) + \frac{1}{E \cdot I} \cdot \frac{w_1 \cdot L_1^{\,3}}{24} + \frac{1}{E \cdot I} \cdot \frac{M_1 \cdot L_1}{6} + \frac{1}{E \cdot I} \cdot \frac{M_2 \cdot L_1}{3} + \frac{\left(y_2 - y_1\right)}{L_1} \qquad (1)$$

Similarly, find the slope at the left end of span 2.
From the **case 8.1.2e**:

$$\theta_{2A1} = \frac{-w_a}{24 \cdot E \cdot I \cdot L_2} \cdot \left(L_2 - a\right)^2 \cdot \left(L_2^{\,2} + 2 \cdot a \cdot L_2 - a^2\right) - \frac{w_L - w_a}{360 \cdot E \cdot I \cdot L_2} \cdot \left(L_2 - a\right)^2 \cdot \left(7 \cdot L_2^{\,2} + 6 \cdot a \cdot L_2 - 3 \cdot a^2\right)$$

where

$$a = 0 \cdot ft$$

$$w_a = w_L = w_1$$

thus

$$\theta_{2A1} = -\frac{1}{E \cdot I} \cdot \frac{w_1 \cdot L_2^{\,3}}{24}$$

From the **case 8.1.3e**:

$$\theta_{2A2} = \frac{-M_2}{6 \cdot E \cdot I \cdot L_2} \cdot \left(2 \cdot L_2^{\,2} - 6a \cdot L_2 + 3 \cdot a^2\right)$$

where

$$a = 0 \cdot ft$$

thus

$$\theta_{2A2} = -\frac{1}{E \cdot I} \cdot \frac{M_2 \cdot L_2}{3}$$

and

$$\theta_{2A3} = \frac{M_3}{6 \cdot E \cdot I \cdot L_2} \cdot \left(2 \cdot L_2^{\,2} - 6a \cdot L_2 + 3 \cdot a^2\right)$$

where

$$a = L_2$$

thus

$$\theta_{2A3} = -\frac{1}{E \cdot I} \cdot \frac{M_3 \cdot L_2}{6}$$

Thus, using the **cases 2e** and **3e** from the **Table 3**, the expression for the slope at the left end of span 2 is

$$\theta_2 = -\frac{1}{E \cdot I} \cdot \frac{w_1 \cdot L_2^{\,3}}{24} - \frac{1}{E \cdot I} \cdot \frac{M_2 \cdot L_2}{3} - \frac{1}{E \cdot I} \cdot \frac{M_3 \cdot L_2}{6} + \frac{\left(y_3 - y_2\right)}{L_2} \qquad (2)$$

Equating the slopes given by the equations 1 & 2 shows their relationship:

$$\frac{1}{E\cdot I}\cdot\frac{W_2\cdot a_1}{6\cdot L_1}\cdot\left(L_1^{\ 2}-a_1^{\ 2}\right)+\frac{1}{E\cdot I}\cdot\frac{w_1\cdot L_1^{\ 3}}{24}+\frac{1}{E\cdot I}\cdot\frac{M_1\cdot L_1}{6}+\frac{1}{E\cdot I}\cdot\frac{M_2\cdot L_1}{3}+\frac{\left(y_2-y_1\right)}{L_1}=-\frac{1}{E\cdot I}\cdot\frac{w_1\cdot L_2^{\ 3}}{24}\ \ldots(3)$$
$$+-\frac{1}{E\cdot I}\cdot\frac{M_2\cdot L_2}{3}\ \ldots$$
$$+-\frac{1}{E\cdot I}\cdot\frac{M_3\cdot L_2}{6}\ \ldots$$
$$+\frac{\left(y_3-y_2\right)}{L_2}$$

Or if $y_2=y_1$ and $y_3=y_2$

$$\frac{W_2\cdot a_1}{6\cdot L_1}\cdot\left(L_1^{\ 2}-a_1^{\ 2}\right)+\frac{w_1\cdot L_1^{\ 3}}{24}+\frac{M_1\cdot L_1}{6}+\frac{M_2\cdot L_1}{3}=-\frac{w_1\cdot L_2^{\ 3}}{24}-\frac{M_2\cdot L_2}{3}-\frac{M_3\cdot L_2}{6}$$

Or

$$\frac{M_2\cdot\left(L_1+L_2\right)}{3}=-\frac{w_1\cdot\left(L_1^{\ 3}+L_2^{\ 3}\right)}{24}-\frac{W_2\cdot a_1}{6\cdot L_1}\cdot\left(L_1^{\ 2}-a_1^{\ 2}\right)-\frac{M_3\cdot L_2}{6}-\frac{M_1\cdot L_1}{6}$$

where the moments M_3 and M_1 equal 0 in our case:

$$M_1 := 0\cdot ft\cdot kip$$

$$M_3 := 0\cdot ft\cdot kip$$

$$M_2(a) := \frac{3}{L_1+L_2}\cdot\left[-\frac{w_1\cdot\left(L_1^{\ 3}+L_2^{\ 3}\right)}{24}-\frac{W_2\cdot a}{6\cdot L_1}\cdot\left(L_1^{\ 2}-a^2\right)-\frac{M_3\cdot L_2}{6}-\frac{M_1\cdot L_1}{6}\right]$$

$$M_2\left(a_1\right) = -1.5\cdot ft\cdot kip$$

Reactions on supports:

$$R_{1A}(a) := \frac{W_2}{L_1}\cdot\left(L_1-a\right)+\frac{w_1\cdot L_1}{2}-\frac{M_1}{L_1}+\frac{M_2(a)}{L_1} \qquad R_{1A}\left(a_1\right) = 0.583\cdot kip$$

$$R_1(a) := R_{1A}(a) \qquad R_1\left(a_1\right) = 0.583\cdot kip$$

$$R_{2B}(a) := \frac{W_2\cdot a}{L_1}+\frac{w_1\cdot L_1}{2}+\frac{M_1}{L_1}-\frac{M_2(a)}{L_1} \qquad R_{2B}\left(a_1\right) = 0.784\cdot kip$$

$$R_{2A}(a) := \frac{w_1\cdot L_2}{2}-\frac{M_2(a)}{L_2}+\frac{M_3}{L_2} \qquad R_{2A}\left(a_1\right) = 0.574\cdot kip$$

$R_2(a) := R_{2B}(a) + R_{2A}(a)$ $R_2(a_1) = 1.358 \cdot kip$

$$R_{3B}(a) := \frac{w_1 \cdot L_2}{2} + \frac{M_2(a)}{L_2} - \frac{M_3}{L_2}$$ $R_{3B}(a_1) = 0.274 \cdot kip$

$R_3(a) := R_{3B}(a)$ $R_3(a_1) = 0.274 \cdot kip$

Checking

$R_1(a_1) + R_2(a_1) + R_3(a_1) = 2.215 \cdot kip$

$w_1 \cdot (L_1 + L_2) + W_2 = 2.215 \cdot kip$

Shear forces

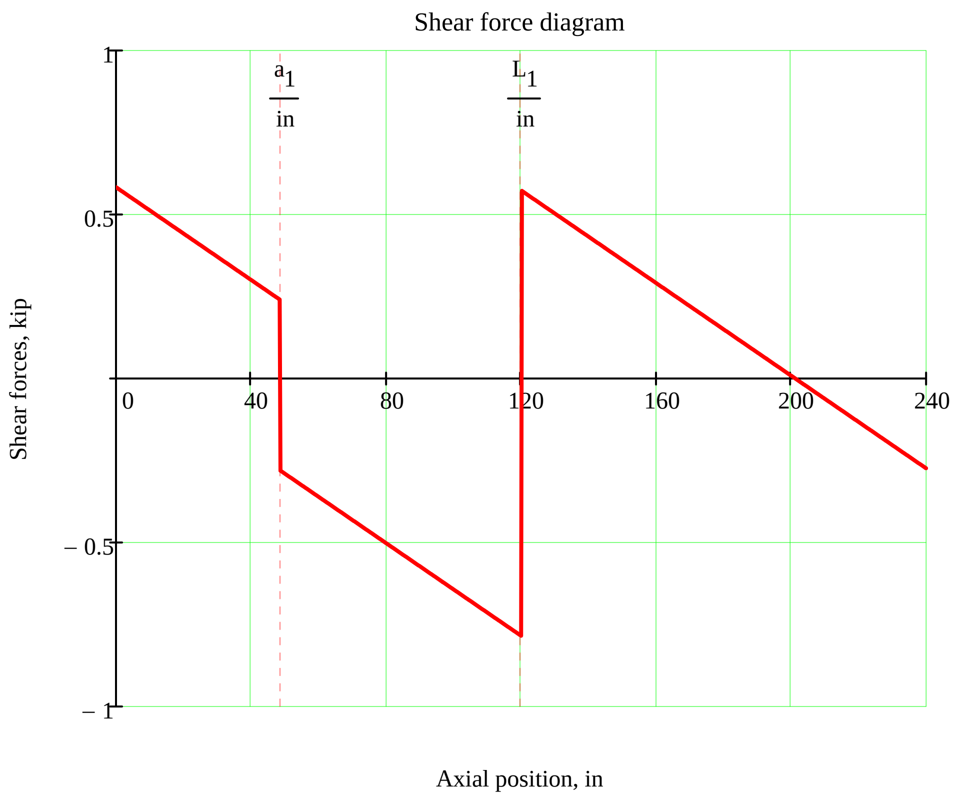
$V(x,a) := -w_1 \cdot x + R_1(a) - W_2 \cdot (x > a) + R_2(a) \cdot (x > L_1)$

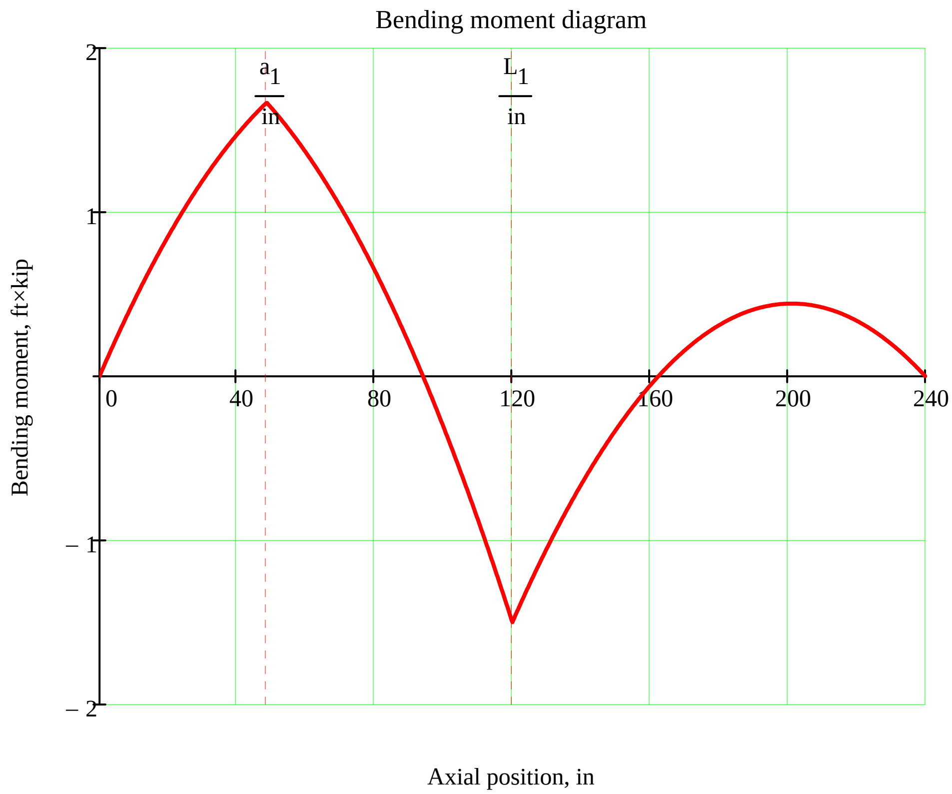
Bending moments

$$M(x,a) := -\frac{w_1 \cdot x^2}{2} + R_1(a) \cdot x - W_2 \cdot (x - a) \cdot (x > a) + R_2(a) \cdot (x - L_1) \cdot (x > L_1)$$

$L := L_1 + L_2$ $L = 240 \cdot in$

$x := 0, 0.001 \cdot L .. L$





$M_{max} := M(a_1, a_1)$ $\quad M_{max} = 1.667 \cdot kip \cdot ft$

$M_{min} := M(L_1, a_1)$ $\quad M_{min} = -1.5 \cdot kip \cdot ft$

$M_{mv} := max(|M_{max}|, |M_{min}|)$ $\quad M_{mv} = 1.667 \cdot kip \cdot ft$

Changing the position a_1 of the load W_2 along the beam we could see that for these boundary conditions the maximum bending moment is at $a_1 = 0.405 \cdot L_1$ and equals to $M_{mv} = 1.667 \cdot kip \cdot ft$ at $x = a_1$.

Now lets select the beam shape. The required section modulus of the beam is:

$S_d := \frac{M_{mv}}{\sigma_{alw}}$ $\quad S_d = 1.6 \cdot in^3$

A hollow rectangular shape has been selected. Section properties for this shape are given in the Table A.1, page 802. This table also has a Mathcad worksheet. Lets copy the equations from this worksheet.

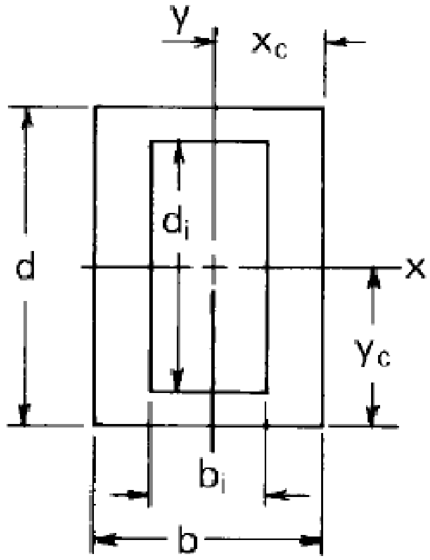


Figure 3

Outer width of the section: $b := 1.75 \cdot in$

Outer depth of the section: $d := 4.5 \cdot in$

Wall thickness: $t := \frac{1}{8} \cdot in$

Inner width of the section: $b_i := b - 2 \cdot t$ $\quad b_i = 1.5 \cdot in$

Inner depth of the section: $d_i := d - 2 \cdot t$ $\quad d_i = 4.25 \cdot in$

Area of cross section:

$A := b \cdot d - b_i \cdot d_i$ $\quad A = 1.5 \cdot in^2$

Distance from the centroid to extremities (refer to **Fig. 3**):

$y_c := \frac{d}{2}$ $\quad y_c = 2.25 \cdot in$

Moment of inertia about central axis X:

$I_x := \frac{b \cdot d^3 - b_i \cdot d_i^3}{12}$ $\quad I_x = 3.693 \cdot in^4$

Section modulus

$S_x := \frac{I_x}{y_c}$ $\quad S_x = 1.641 \cdot in^3$

Changing the dimensions **b, d,** and **t** gives the beam RT $1\frac{3}{4} \times 4\frac{1}{2} \times \frac{1}{8}$ in.

Additional User Notices